CS 154, Lecture 3:
DFA ≡ NFA,
Regular Expressions
Homework 1 is coming out ...
Deterministic Finite Automata

Computation with finite memory
Non-Deterministic Finite Automata

Computation with finite memory
and “verified guessing”
From NFAs to DFAs

Input: NFA $N = (Q, \Sigma, \delta, Q_0, F)$

Output: DFA $M = (Q', \Sigma, \delta', q_0', F')$

To learn if an NFA accepts, we could do the computation in parallel, maintaining the set of all possible states that can be reached.

Idea:
Set $Q' = 2^Q$
From NFAs to DFAs: **Subset Construction**

Input: NFA $N = (Q, \Sigma, \delta, Q_0, F)$

Output: DFA $M = (Q', \Sigma, \delta', q_0', F')$

\[
Q' = 2^Q \\
\delta' : Q' \times \Sigma \to Q' \\
\delta'(R, \sigma) = \bigcup_{r \in R} \varepsilon(\delta(r, \sigma)) \quad \text{(*)} \\
q_0' = \varepsilon(Q_0) \\
F' = \{ R \in Q' : \text{f \in R for some f \in F} \} \\
\]

(*) For $S \subseteq Q$, the $\varepsilon$-closure of $S$ is

\[
\varepsilon(S) = \{ q : q \text{ reachable from some } s \in S \text{ by taking 0 or more } \varepsilon \text{ transitions} \} 
\]
Example of the $\epsilon$-closure

\[ \epsilon(\{q_0\}) = \{q_0, q_1, q_2\} \]
\[ \epsilon(\{q_1\}) = \{q_1, q_2\} \]
\[ \epsilon(\{q_2\}) = \{q_2\} \]
Given: NFA $N = (\{1,2,3\}, \{a,b\}, \delta, \{1\}, \{1\})$

Construct: Equivalent DFA $M = (2^{\{1,2,3\}}, \{a,b\}, \delta', \{1,3\}, \ldots)$

$\varepsilon(\{1\}) = \{1,3\}$

$\varepsilon(\{1\}) = \{1,3\}$
Reverse Theorem for Regular Languages

Theorem: The reverse of a regular language is also a regular language

If a language can be recognized by a DFA that reads strings from right to left, then there is an “normal” DFA that accepts the same language

Proof?

Given a DFA for a language L, “reverse” its arrows and flip its start and accept states, getting an NFA. Convert that NFA back to a DFA!
Using NFAs in place of DFAs can make proofs about regular languages much easier!

Remember this on homework/exams!
Union Theorem using NFAs?
Regular Languages are closed under concatenation

**Concatenation:** \( A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \} \)

Given DFAs \( M_1 \) and \( M_2 \), connect the accept states of \( M_1 \) to the start states of \( M_2 \)

\[ L(N) = L(M_1) \cdot L(M_2) \]
Regular Languages are closed under star

\[ A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \} \]

Let \( M \) be a DFA, and let \( L = L(M) \)

We can construct an NFA \( N \) that recognizes \( L^* \)
Formally, the construction is:

Input: DFA $M = (Q, \Sigma, \delta, q_1, F)$

Output: NFA $N = (Q', \Sigma, \delta', q_0, F')$

$$Q' = Q \cup \{q_0\}$$

$$F' = F \cup \{q_0\}$$

$$\delta'(q,a) = \begin{cases} 
\{\delta(q,a)\} & \text{if } q \in Q \text{ and } a \neq \varepsilon \\
\{q_1\} & \text{if } q \in F \text{ and } a = \varepsilon \\
\{q_1\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\
\emptyset & \text{if } q = q_0 \text{ and } a \neq \varepsilon \\
\emptyset & \text{else}
\end{cases}$$
Regular Languages are Closed Under Star

How would we prove that this NFA construction works?

Want to show: \( L(N) = L^* \)

1. \( L(N) \supseteq L^* \)
2. \( L(N) \subseteq L^* \)
1. $L(N) \supseteq L^*$

Assume $w = w_1...w_k$ is in $L^*$ where $w_1,...,w_k \in L$

We show $N$ accepts $w$ by induction on $k$

**Base Cases:**

- $k = 0$ \quad (w = \varepsilon)
- $k = 1$ \quad (w \in L)

**Inductive Step:**

Assume $N$ accepts all strings $v = v_1...v_k \in L^*$, $v_i \in L$

Let $u = u_1...u_ku_{k+1} \in L^*$, $u_j \in L$

Since $N$ accepts $u_1...u_k$ (by induction) and $M$ accepts $u_{k+1}$, $N$ also accepts $u$ (by construction)
Assume $w$ is accepted by $N$; we want to show $w \in L^*$

If $w = \varepsilon$, then $w \in L^*$

I.H. $N$ accepts $u$ and takes at most $k \varepsilon$-transitions

$\Rightarrow u \in L^*$

Let $w$ be accepted by $N$ with $k+1 \varepsilon$-transitions.

Write $w$ as $w=uv$, where $v$ is the substring read after the last $\varepsilon$-transition

$u \in L(N)$, so by I.H.

$u \in L^*$

$v \in L$

$w = uv \in L^*$
Closure Properties for Regular Languages

√ **Union:** \( A \cup B = \{ w \mid w \in A \text{ or } w \in B \} \)

√ **Intersection:** \( A \cap B = \{ w \mid w \in A \text{ and } w \in B \} \)

√ **Complement:** \( \overline{A} = \{ w \in \Sigma^* \mid w \notin A \} \)

√ **Reverse:** \( A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A, \ w_i \in \Sigma \} \)

√ **Concatenation:** \( A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \} \)

√ **Star:** \( A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \} \)

**Theorem:** if \( A \) and \( B \) are regular then so are: \( A \cup B, A \cap B, \overline{A}, A^R, A \cdot B, \) and \( A^* \)
Regular Expressions

Computation as simple, logical description

A totally different way of thinking about computation: *What is the complexity of describing the strings in the language?*
Inductive Definition of Regexp

Let $\Sigma$ be an alphabet. We define the regular expressions over $\Sigma$ inductively:

For all $\sigma \in \Sigma$, $\sigma$ is a regexp

$\varepsilon$ is a regexp

$\emptyset$ is a regexp

If $R_1$ and $R_2$ are both regexps, then

$(R_1R_2)$, $(R_1 + R_2)$, and $(R_1)^*$ are regexps
Precedence Order:

\[
\ast \quad \text{then} \quad \cdot \quad \text{then} \quad +
\]

Example: \( R_1 \ast R_2 + R_3 = ((R_1 \ast) \cdot R_2) + R_3 \)
Definition: Regexps Represent Languages

The regexp $\sigma \in \Sigma$ represents the language $\{\sigma\}$
The regexp $\varepsilon$ represents $\{\varepsilon\}$
The regexp $\emptyset$ represents $\emptyset$

If $R_1$ and $R_2$ are regular expressions representing $L_1$ and $L_2$ then:

$(R_1 R_2)$ represents $L_1 \cdot L_2$
$(R_1 + R_2)$ represents $L_1 \cup L_2$
$(R_1)^*$ represents $L_1^*$
Regexps Represent Languages

For every regexp $R$, define $L(R)$ to be the language that $R$ represents.

A string $w \in \Sigma^*$ is accepted by $R$ (or, $w$ matches $R$) if $w \in L(R)$.

Examples: 0, 010, and 01010 match $(01)^*0$
11010110100100 matches $(0+1)^*0$
Assume $\Sigma = \{0,1\}$

{ $w$ | $w$ has exactly a single 1 }

$0^*10^*$

{ $w$ | $w$ contains 001 }

$(0+1)^*001(0+1)^*$
Assume $\Sigma = \{0, 1\}$

What language does the regexp $\emptyset^*$ represent?

$\{\varepsilon\}$
Assume $\Sigma = \{0,1\}$

\{ w | w has length $\geq 3$ and its 3rd symbol is 0 \}

$(0+1)(0+1)0(0+1)^*$
Assume $\Sigma = \{0,1\}$

$\{w \mid \text{every odd position in } w \text{ is a 1} \}$

$(1(0 + 1))^*(1 + \varepsilon)$
Assume $\Sigma = \{0, 1\}$

\[
\{ \mathbf{w} \mid \text{w has equal number of occurrences of 01 and 10} \}
\]

= \[
\{ \mathbf{w} \mid \mathbf{w} = 1, \mathbf{w} = 0, \text{ or } \mathbf{w} = \varepsilon, \text{ or }
\begin{align*}
&\text{w starts with a 0 and ends with a 0, or} \\
&\text{w starts with a 1 and ends with a 1} \}
\]

Claim:
A string $\mathbf{w}$ has equal occurrences of 01 and 10

$\iff$ $\mathbf{w}$ starts and ends with the same bit.

\[1 + 0 + \varepsilon + 0(0+1)^*0 + 1(0+1)^*1\]
DFAs $\equiv$ NFAs $\equiv$ Regular Expressions!

$L$ can be represented by some regexp

$\iff$ $L$ is regular
L can be represented by some regexp

⇒ L is regular
Given any regexp $R$, we will construct an NFA $N$ s.t. $N$ accepts exactly the strings accepted by $R$.

**Proof by induction on the length of the regexp $R$.**

**Base Cases** ($R$ has length 1):
- Given any regexp $R$, we will construct an NFA $N$ s.t. $N$ accepts exactly the strings accepted by $R$.
- $R = \sigma$:

$$
\begin{array}{c}
R = \sigma \\
\Rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
$$

- $R = \varepsilon$:

$$
\begin{array}{c}
R = \varepsilon \\
\Rightarrow \\
\rightarrow \\
\end{array}
$$

- $R = \emptyset$:

$$
\begin{array}{c}
R = \emptyset \\
\Rightarrow \\
\rightarrow \\
\end{array}
$$

$L$ can be represented by some regexp

$\Rightarrow L$ is regular
Induction Step: Suppose every regexp of length < k represents some regular language.

Consider a regexp R of length k > 1

Three possibilities for R:

\[ R = R_1 + R_2 \]

\[ R = R_1 R_2 \]

\[ R = (R_1)^* \]
Induction Step: Suppose every regexp of length $< k$ represents some regular language.

Consider a regexp $R$ of length $k > 1$

Three possibilities for $R$:

- $R = R_1 + R_2$  
  By induction, $R_1$ and $R_2$ represent some regular languages, $L_1$ and $L_2$

- $R = R_1 R_2$  
  But $L(R) = L(R_1 + R_2) = L_1 \cup L_2$

- $R = (R_1)^*$  
  so $L(R)$ is regular, by the union theorem!
Induction Step: Suppose every regexp of length $< k$ represents some regular language.

Consider a regexp $R$ of length $k > 1$

Three possibilities for $R$:

$$R = R_1 + R_2$$  By induction, $R_1$ and $R_2$ represent some regular languages, $L_1$ and $L_2$

$$R = R_1 \cdot R_2$$  But $L(R) = L(R_1 \cdot R_2) = L_1 \cdot L_2$

$$R = (R_1)^*$$  so $L(R)$ is regular by the concatenation theorem
Induction Step: Suppose every regexp of length \(< k\) represents some regular language.

Consider a regexp \(R\) of length \(k \geq 1\)

Three possibilities for \(R\):

\[
R = R_1 + R_2 \quad \text{By induction, } R_1 \text{ and } R_2 \text{ represent some regular languages, } L_1 \text{ and } L_2
\]

\[
R = R_1 R_2 \quad \text{But } L(R) = L(R_1^*) = L_1^* \text{ so } L(R) \text{ is regular, by the star theorem}
\]

\[
R = (R_1)^* \quad \text{so } L(R) \text{ is regular, by the star theorem}
\]
Induction Step: Suppose every regexp of length $< k$ represents some regular language.

Consider a regexp $R$ of length $k > 1$

Three possibilities for $R$:

- $R = R_1 + R_2$ By induction, $R_1$ and $R_2$ represent some regular languages, $L_1$ and $L_2$
- $R = R_1 R_2$ But $L(R) = L(R_1^*) = L_1^*$ so $L(R)$ is regular, by the star theorem
- $R = (R_1)^*$

Therefore: If $L$ is represented by a regexp, then $L$ is regular
Give an NFA that accepts the language represented by \((1(0+1))^*\)

Regular expression: \((1(0+1))^*\)
L can be represented by a regexp

L is a regular language

Idea: Transform an NFA for L into a regular expression by removing states and re-labeling the arcs with regular expressions

Rather than reading in just letters from the string on a step, we can read in entire substrings.
This GNFA recognizes $L(a^*b(cb)^*a)$

Is $aaabcbcba$ accepted or rejected?
Is $bba$ accepted or rejected?
Is $bcba$ accepted or rejected?

This GNFA recognizes $L(a^*b(cb)^*a)$
Add unique start and accept states
While the machine has more than 2 states:

Pick an internal state, rip it out and re-label the arrows with regexps, to account for paths through the missing state.
In general:

$$R(q_1, q_2)R(q_2, q_2)^*R(q_2, q_3)^*R(q_1, q_3)$$

While the machine has more than 2 states:

**In general:**

While the machine has more than 2 states:
\[
R(q_0,q_3) = (a^*b)(a+b)^* \text{ represents } L(N)
\]
DFAs \leftrightarrow NFAs

\downarrow\quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow

\text{Regular Languages} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad //
Parting thoughts:
Regular Languages can be defined by their closure properties
NFA=DFA, does it mean that non-determinism is free for Finite Automata?

Questions?