Reverse Theorem for Regular Languages

Theorem: The reverse of a regular language is also a regular language

If a language can be recognized by a DFA that reads strings from right to left, then there is an “normal” DFA that accepts the same language

Proof?

Given a DFA for a language L, “reverse” its arrows and flip its start and accept states, getting an NFA. Convert that NFA back to a DFA!
Using NFAs in place of DFAs can make proofs about regular languages much easier!

Remember this on homework/exams!
Union Theorem using NFAs?
Regular Languages are closed under concatenation

**Concatenation:** \( A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \} \)

Given DFAs \( M_1 \) and \( M_2 \), connect the accept states of \( M_1 \) to the start states of \( M_2 \)

\[
L(N) = L(M_1) \cdot L(M_2)
\]
Regular Languages are closed under star

\[ A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \} \]

Let \( M \) be a DFA, and let \( L = L(M) \)

We can construct an NFA \( N \) that recognizes \( L^* \)
Formally, the construction is:

Input: DFA $M = (Q, \Sigma, \delta, q_1, F)$

Output: NFA $N = (Q', \Sigma, \delta', q_0, F')$

$$Q' = Q \cup \{q_0\}$$

$$F' = F \cup \{q_0\}$$

$$\delta'(q,a) = \begin{cases} 
\{\delta(q,a)\} & \text{if } q \in Q \text{ and } a \neq \varepsilon \\
\{q_1\} & \text{if } q \in F \text{ and } a = \varepsilon \\
\{q_1\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\
\emptyset & \text{if } q = q_0 \text{ and } a \neq \varepsilon \\
\emptyset & \text{else}
\end{cases}$$
Regular Languages are Closed Under Star

How would we prove that this NFA construction works?

Want to show: \( L(N) = L^* \)

1. \( L(N) \supseteq L^* \)

2. \( L(N) \subseteq L^* \)
1. $L(N) \supseteq L^*$

Assume $w = w_1...w_k$ is in $L^*$ where $w_1,...,w_k \in L$

We show $N$ accepts $w$ by induction on $k$

**Base Cases:**

✓ $k = 0$ \hspace{1cm} ($w = \varepsilon$)
✓ $k = 1$ \hspace{1cm} ($w \in L$)

**Inductive Step:**

Assume $N$ accepts all strings $v = v_1...v_k \in L^*$, $v_i \in L$

Let $u = u_1...u_k u_{k+1} \in L^*$, $u_j \in L$

Since $N$ accepts $u_1...u_k$ (by induction) and $M$ accepts $u_{k+1}$, $N$ also accepts $u$ (by construction)
Assume $w$ is accepted by $N$; we want to show $w \in L^*$.

If $w = \varepsilon$, then $w \in L^*$.

I.H. $N$ accepts $u$ and takes at most $k \varepsilon$-transitions.

$\Rightarrow u \in L^*$

Let $w$ be accepted by $N$ with $k+1 \varepsilon$-transitions.

Write $w$ as $w = uv$, where $v$ is the substring read after the last $\varepsilon$-transition.

$w = uv \in L^*$
Closure Properties for Regular Languages

- **Union**: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$
- **Intersection**: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$
- **Complement**: $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$
- **Reverse**: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A, w_i \in \Sigma \}$
- **Concatenation**: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$
- **Star**: $A^* = \{ s_1 \ldots s_k \mid k \geq 0 \text{ and each } s_i \in A \}$

**Theorem**: if $A$ and $B$ are regular then so are:
- $A \cup B$, $A \cap B$, $\neg A$, $A^R$, $A \cdot B$, and $A^*$