Limitations on DFAs (I):

Pumping Lemma

CS 154, Omer Reingold
Non-Regular Languages

Regular or Not?

\[ D = \{ w \mid w \text{ has equal number of occurrences of } 01 \text{ and } 10 \} \]

REGULAR!

\[ C = \{ w \mid w \text{ has equal number of } 1s \text{ and } 0s \} \]

NOT REGULAR!

How can we prove that there is no DFA for a particular language?

- Surprising Algorithms (even in restricted models) are routinely being discovered
The Pumping Lemma: Structure in Regular Languages

Let $L$ be a regular language

Then there is a positive integer $P$ s.t.

For all strings $w \in L$ with $|w| \geq P$ there is a way to write $w = xyz$, where:

1. $|y| > 0$ (that is, $y \neq \varepsilon$)
2. $|xy| \leq P$
3. For all $i \geq 0$, $xy^iz \in L$

Why is it called the pumping lemma?

The word $w$ gets pumped into longer and longer strings...
Proof: Let $M$ be a DFA that recognizes $L$

Let $P$ be the number of states in $M$

Let $w$ be a string where $w \in L$ and $|w| \geq P$

We show: $w = xyz$

1. $|y| > 0$
2. $|xy| \leq P$
3. $xy^iz \in L$ for all $i \geq 0$

Claim: There must exist $j$ and $k$ such that $0 \leq j < k \leq P$, and $q_j = q_k$
Generalized Pumping Lemma:

Let $L$ be a regular language

Then there is a positive integer $P$ s.t.

for all strings $awb \in L$ with $|w| \geq P$ there is a way to write $w = xyz$, where:

1. $|y| > 0$ (that is, $y \neq \varepsilon$)
2. $|xy| \leq P$
3. For all $i \geq 0$, $axy^izb \in L$
Let’s prove that \( \text{EQ} = \{ w \mid \#1s = \#0s \} \) is not regular.

By contradiction. Assume \( \text{EQ} \) is regular. Let \( P \) be as in pumping lemma. Let \( w = 0^P 1^P \in \text{EQ} \).

\[ \Rightarrow \] Can write \( w = xyz \), with \( |y| > 0, |xy| \leq P \), such that for all \( i \geq 0 \), \( xy^i z \) is also in \( \text{EQ} \).

Claim: The string \( y \) must be all zeroes.

Why? Because \( |xy| \leq P \) and \( w = xyz = 0^P 1^P \)

But then \( xyyz \) has more 0s than 1s. Contradiction!
Applying the Pumping Lemma

Prove: \( SQ = \{0^n^2 \mid n \geq 0\} \) is not regular

Assume \( SQ \) is regular. Let \( w = 0^{p^2} \)

\[ \Rightarrow \text{Can write } w = xyz, \text{ with } |y| > 0, \ |xy| \leq P, \] such that for all \( i \geq 0, \ xy^iz \) is also in \( SQ \)

So \( xyyz \in SQ \). Note that \( xyyz = 0^{p^2+|y|} \)

Note that \( 0 < |y| < P \)

So \( |xyyz| = p^2 + |y| \leq p^2 + P < p^2 + 2P + 1 = (p+1)^2 \)

and \( p^2 < |xyyz| < (p+1)^2 \)

Therefore \( |xyyz| \) is not a perfect square!

Hence \( 0^{p^2+|y|} = xyyz \notin SQ \), so our assumption must be false.

\[ \Rightarrow SQ \text{ is not regular!} \]
Parting thoughts:
Pumping for contradictions
DFAs can’t count