Communication Complexity
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A model capturing one aspect of distributed computing.

Here focus on two parties: Alice and Bob

Function $f : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$

Two inputs, $x \in \{0,1\}^*$ and $y \in \{0,1\}^*$

We assume $|x| = |y| = n$, Think of $n$ as HUGE

Alice only knows $x$, Bob only knows $y$

Goal: Compute $f(x, y)$ by communicating as few bits as possible between Alice and Bob

We do not count computation cost. We only care about the number of bits communicated.
Alice and Bob Have a Conversation

In every step: A bit is sent, which is a function of the party’s input and all the bits communicated so far.

Communication cost = number of bits communicated = 4 (in the example)
We assume Alice and Bob alternate in communicating, and the last bit sent is $f(x,y)$

More sophisticated models: separate number of rounds from number of bits
Def. A *protocol* for a function $f$ is a pair of functions $A, B : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1, \text{STOP}\}$ with the semantics:

On input $(x, y)$, let $r := 0$, $b_0 = \varepsilon$

While $(b_r \neq \text{STOP})$,

$r++$

If $r$ is odd, Alice sends $b_r = A(x, b_1 \ldots b_{r-1})$

else Bob sends $b_r = B(y, b_1 \ldots b_{r-1})$

Output $b_{r-1}$.

Number of *rounds* $= r - 1$
Def. The cost of a protocol $P$ for $f$ on $n$-bit strings is

$$\max_{x,y \in \{0,1\}^n} \text{number of rounds in } P \text{ to compute } f(x, y)$$

The communication complexity of $f$ on $n$-bit strings is the minimum cost over all protocols for $f$ on $n$-bit strings = the minimum number of rounds used by any protocol that computes $f(x, y)$, over all $n$-bit $x, y$
Example. Let $f : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$ be arbitrary.

There is always a “trivial” protocol:
Alice sends the bits of her $x$ in odd rounds
Bob sends the bits of his $y$ in even rounds
After $2n$ rounds, they both know each other’s input!

The communication complexity of every $f$ is at most $2n$. 
Example: \( \text{PARITY}(x, y) = \sum_i x_i + \sum_i y_i \mod 2 \).

What’s a good protocol for computing PARITY?

Alice sends \( b_1 = (\sum_i x_i \mod 2) \)
Bob sends \( b_2 = (b_1 + \sum_i y_i \mod 2) \). Alice stops.

The communication complexity of PARITY is 2
Example: MAJORITY($x, y$) = most frequent bit in $xy$

What’s a good protocol for computing MAJORITY?

Alice sends $N_x = \text{number of 1s in } x$
Bob computes $N_y = \text{number of 1s in } y$, sends 1 iff $N_x + N_y$ is greater than $(|x| + |y|)/2 = n$

Communication complexity of MAJORITY is $O(\log n)$
Example: \text{EQUALS}(x, y) = 1 \iff x = y

What’s a good protocol for computing \text{EQUALS} ???

\text{Communication complexity of} \ \text{EQUALS is at most} \ 2n
Connection to Streaming and DFAs

Let $L \subseteq \{0,1\}^*$
Def. $f_L: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$
for $x, y$ with $|x| = |y|$ as:
\[
f_L(x, y) = 1 \iff xy \in L
\]

Examples:
$L = \{ x \mid x \text{ has an odd number of 1s} \}$
\[
\Rightarrow f_L(x, y) = \text{PARITY}(x,y) = \sum_i x_i + \sum_i y_i \mod 2
\]
$L = \{ x \mid x \text{ has more 1s than 0s} \}$
\[
\Rightarrow f_L(x, y) = \text{MAJORITY}(x,y)
\]
$L = \{ xx \mid x \in \{0,1\}^* \}$
\[
\Rightarrow f_L(x, y) = \text{EQUALS}(x,y)
\]
Theorem: If $L$ has a streaming algorithm using $\leq s$ space, then the comm. complexity of $f_L$ is at most $O(s)$.

Proof: Alice runs streaming algorithm $A$ on $x$. Sends the memory content of $A$: this is $s$ bits of space. Bob starts up $A$ with that memory content, runs $A$ on $y$. Gets an output bit, sends to Alice.
Connection to Streaming and DFAs

Let $L \subseteq \{0,1\}^*$  Def. $f_L(x, y) = 1 \iff xy \in L$

Theorem: If $L$ has a streaming algorithm using $\leq s$ space, then the comm. complexity of $f_L$ is at most $O(s)$.

Corollary: For every regular language $L$, the comm. complexity of $f_L$ is $O(1)$.

Example: Comm. Complexity of PARITY is $O(1)$

Corollary: Comm. Complexity of MAJORITY is $O(\log n)$, because there’s a streaming algorithm for

$\{x : x \text{ has more 1’s than 0’s}\}$ with $O(\log n)$ space

What about the Comm. Complexity of EQUALS?
Communication Complexity of EQUALS

Theorem: The comm. complexity of EQUALS is $\Theta(n)$. In particular, every protocol for EQUALS needs $\geq n$ bits of communication.

No communication protocol can do much better than “send your whole input”!

Corollary: $L = \{ww \mid w \text{ in } \{0,1\}^*\}$ is not regular.

Moreover, every streaming algorithm for $L$ needs $cn$ bits of memory, for some constant $c > 0$. 
Theorem: The comm. complexity of EQUALS is $\Theta(n)$. In particular, every protocol for EQUALS needs $\geq n$ bits of communication.

Idea: Consider all possible ways A & B can communicate.

Definition: The communication pattern of a protocol on inputs $(x, y)$ is the sequence of bits that Alice & Bob send.

Pattern: 0110
Theorem: The communication complexity of EQUALS is $\Theta(n)$. In particular, every protocol for EQUALS needs $\geq n$ bits of communication.

Proof: By contradiction. Suppose CC of EQUALS is $\leq n - 1$. Then there are $\leq 2^{n-1}$ possible communication patterns of that protocol, over all pairs of inputs $(x, y)$ with $n$ bits each.

Claim: There are $x \neq y$ such that on $(x, x)$ and on $(y, y)$, the protocol uses the same pattern $P$.

Now, what is the communication pattern on $(x, y)$?
This pattern is also $P$ (WHY?)
So Alice & Bob output the same bit on $(x, y)$ and $(x, x)$.
But $\text{EQUALS}(x, y) = 0$ and $\text{EQUALS}(x, x) = 1$. Contradiction!
Randomized Protocols Help!

EQUALS needs $c\ n$ bits of communication, but...

Theorem: For all $(x, y)$ of $n$ bits each, there is a randomized protocol for $EQUALS(x, y)$ using only $O(\log n)$ bits of communication, which works with probability 99.9%!

Use Error Correcting Codes … E.g:
- Alice picks a random prime number $p$ between 2 and $n^2$.
- She sends $p$ and her string $x$ modulo $p$.
- This is a number between 0 and $n^2$, takes $O(\log n)$ bits to send.
- Bob checks whether $y = x$ modulo $p$. Sends output bit.
Why does it work (with high probability)?
Parting thoughts:
Communication Complexity: Powerful Tool (we seen just a tiny demonstration).
Communication Complexity, Streaming Algorithms and Regular languages – connected.
Randomness – could be a useful resource of computation (II)